A Conception of Inductive Logic: Example

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This paper presents a simple example of inductive logic done according to the conception of inductive logic described in Maher (2006). Thus it defines a function p that is intended to be an explicatum for inductive probability.

1. The language $\mathcal{L}$

The arguments of p will be sentences of a formal language $\mathcal{L}$ that is defined by the following clauses.[1]

1. $\mathcal{L}$ contains the following signs:
   a) Individual constants: 'a' and 'b'
   b) Predicates: 'F' and 'G'
   c) Connectives: '$\sim$', '$\vee$', and '.'
   d) Parentheses: '(' and ')'
2. The following rules specify which expressions are sentences in $\mathcal{L}$:
   a) If ϕ is a predicate and α an individual constant then $\phi\alpha$ is a sentence.
   b) If σ is a sentence then so is $\sim\sigma$.
   c) If σ_1 and σ_2 are sentences then so are $(\sigma_1 \vee \sigma_2)$ and $(\sigma_1.\sigma_2)$.
   d) Nothing is a sentence unless its being so follows from a)–c).
3. Let there be an urn that at time t contains two balls, one of which is heavier than the other. The meaning of the individual constants and predicates in $\mathcal{L}$ is given by the following rules.
   a) 'a' designates the lighter of the two balls in the urn.
   b) 'b' designates the heavier of the two balls in the urn.

[1] In order to describe this language I will use the following standard conventions: (a) An expression in single quotation marks designates the expression contained by the quotation marks. (b) Signs of $\mathcal{L}$ designate themselves. (c) A concatenation of signs that designate expressions of $\mathcal{L}$ designates the expression obtained by replacing each sign by the expression it designates.

c) 'F' designates the property of being white at time t.

d) 'G' designates the property of being black at time t.

4. Truth in $\mathcal{L}$ is defined by the following rules.

a) If ϕ is a predicate and α is an individual constant then $\phi\alpha$ is true iff the individual that α designates has the property that ϕ designates.

b) If σ is a sentence then $\sim\sigma$ is true iff σ is not true.

c) If σ_1 and σ_2 are sentences then

i) $(\sigma_1 \vee \sigma_2)$ is true iff at least one of σ_1 and σ_2 is true.

ii) $(\sigma_1.\sigma_2)$ is true iff σ_1 and σ_2 are both true.

2. L-concepts

The atomic sentences of $\mathcal{L}$ are 'Fa', 'Ga', 'Fb', and 'Gb'. Let a *model* for $\mathcal{L}$ be an assignment of truth values to each of these atomic sentences. There are $2^4 = 16$ models for $\mathcal{L}$. If M is a model for $\mathcal{L}$ and σ is a sentence in $\mathcal{L}$ then σ is said to be *true in* M if the truth value assignments of M, together with the rules of truth for the connectives of $\mathcal{L}$, imply that σ is true. For example, '$(Fa \vee Fb)$' is true in the models that assign truth to at least one of 'Fa' and 'Fb'.

Let a *logically true sentence* be a sentence that is true in virtue of meanings alone. Since it is logically true that nothing can be black and white at the same time, '$\sim(Fa.Ga)$' and '$\sim(Fb.Gb)$' are logically true sentences in $\mathcal{L}$. Let an *admissible model* for $\mathcal{L}$ be a model in which these sentences are true. There are $3^2 = 9$ admissible models for $\mathcal{L}$. A sentence in $\mathcal{L}$ will be said to be *L-true* if it is true in all admissible models for $\mathcal{L}$. For example, '$(Fa\vee\sim Fa)$' and '$(\sim Fa\vee\sim Ga)$' are L-true. The concept of L-truth is an explicatum for the vague ordinary concept of logical truth.

A sentence in $\mathcal{L}$ will be said to be *L-false* if it is not true in any admissible models. A sentence σ_1 in $\mathcal{L}$ will be said to *L-imply* a sentence σ_2 in $\mathcal{L}$ if the set of admissible models in which σ_1 is true is a subset of the set of admissible models in which σ_2 is true. Sentences σ_1 and σ_2 will be said to be *L-equivalent* if the set of admissible models in which σ_1 is true is the same as the set of admissible models in which σ_2 is true. These concepts are explicata for ordinary concepts of logical falsity, logical implication, and logical equivalence, respectively.

3. The function p

I will now state conditions on the function p that together will serve to define p. To begin, let 'σ_*' designate the sentence '$((Fa \vee Ga).(Fb \vee Gb))$'. The domain of p is specified by:

CONDITION 1. *$p(\sigma_1, \sigma_2)$ exists iff σ_1 and σ_2 are sentences in $\mathcal{L}$ and σ_2 L-implies σ_*.*

The requirement that σ_2 L-imply σ_* is imposed because it makes it easier to define a satisfactory p without entirely trivializing the problem.

The requirement that p satisfy the mathematical laws of conditional probability is here given the following specific form. (In what follows, outer parentheses are dropped in designations of sentences in $\mathcal{L}$, so e.g., '$\sigma_1.\sigma_2$' is to be understood as referring to $(\sigma_1.\sigma_2)$.)

CONDITION 2. *If σ_1, σ_2, and σ_3 are sentences in $\mathcal{L}$ and σ_3 L-implies σ_* then:*

A1. $p(\sigma_1, \sigma_3) \geq 0$.

A2. $p(\sigma_3, \sigma_3) = 1$.

A3. $p(\sigma_1, \sigma_3) + p(\sim\sigma_1, \sigma_3) = 1$, *provided σ_3 is not L-false.*

A4. $p(\sigma_1.\sigma_2, \sigma_3) = p(\sigma_1, \sigma_3)p(\sigma_2, \sigma_1.\sigma_3)$.

A5. If σ_1' and σ_3' are L-equivalent to σ_1 and σ_3 respectively, then $p(\sigma_1', \sigma_3') = p(\sigma_1, \sigma_3)$.

Maher (2004) stated essentially these axioms and deduced many of their consequences, including the usual elementary theorems of probability.

CONDITION 3. $p(Fa, \sigma_*) = p(Fb, \sigma_*) = 1/2$.

The inductive probability of Fa given σ_* is $1/2$, and likewise for Fb. Condition 3 ensures that p agrees with inductive probability in this respect.

CONDITION 4. $p(Fb, Fa.\sigma_*) = 2/3$.

The inductive probability of Fb given $Fa.\sigma_*$ does not have a precise numerical value but it is greater than $1/2$ since Fa increases the inductive probability of Fb. The value of $2/3$ is chosen as a precise value that is convenient and sufficiently similar to the vague inductive probability.

There is one and only one function p that satisfies Conditions 1–4, as I show in the next section. Hence the definition of p is now complete.

4. Proof of the existence and uniqueness of p

Section 4.1 states three new conditions and shows that exactly one function p satisfies them and Condition 1. Sections 4.3 and 4.4 show that these new conditions are equivalent to Conditions 2–4. It follows that exactly one function p satisfies Conditions 1–4.

4.1. Three new conditions

DEFINITION 1. *σ_{ab} = 'Fa.Fb', σ_a = 'Fa.Gb', σ_b = 'Ga.Fb', and $\sigma_\emptyset$ = 'Ga.Gb'.*

CONDITION 5. *$p(\sigma_{ab}, \sigma_*) = 1/3$, $p(\sigma_a, \sigma_*) = 1/6$, $p(\sigma_b, \sigma_*) = 1/6$, and $p(\sigma_\emptyset, \sigma_*) = 1/3$.*

DEFINITION 2. *If σ_1 is any sentence in $\mathcal{L}$ then $r(\sigma_1)$ is the set of all $\sigma_\alpha \in \{\sigma_{ab}, \sigma_a, \sigma_b, \sigma_\emptyset\}$ such that $\sigma_1.\sigma_\alpha$ is not L-false.*

The set $r(\sigma_1)$ is like what Carnap (1950) called the range of σ_1 except that it is relative to σ_*. Note that $r(\sigma_1) = \emptyset$ (the empty set) iff $\sigma_1.\sigma_*$ is L-false.

CONDITION 6. *If σ_1 is a sentence in $\mathcal{L}$ other than σ_{ab}, σ_a, σ_b, or $\sigma_\emptyset$ then*

$$p(\sigma_1, \sigma_*) = \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_*).$$

CONDITION 7. *If σ_1 and σ_3 are sentences in $\mathcal{L}$, σ_3 is not σ_*, and σ_3 L-implies σ_* then*

$$p(\sigma_1, \sigma_3) = \left\{ \begin{array}{cc} \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)} & \text{if } p(\sigma_3, \sigma_*) \neq 0 \\ 1 & \text{if } p(\sigma_3, \sigma_*) = 0 \end{array} \right\}.$$

PROPOSITION 1. *Exactly one function p satisfies Conditions 1 and 5–7.*

Proof. Conditions 5–7 each assign values to $p(\sigma_1, \sigma_3)$ for different pairs $< \sigma_1, \sigma_3 >$, so these conditions are consistent with one another. Furthermore, these three conditions specify a value of $p(\sigma_1, \sigma_3)$ iff σ_1 and σ_3 are sentences in $\mathcal{L}$ and σ_3 L-implies σ_*, so they are also consistent with Condition 1. Hence there exists a function that satisfies Condition 1 and Conditions 5–7. Since Conditions 5–7 specify a unique value of $p(\sigma_1, \sigma_3)$ for all such σ_1 and σ_3, there is only one function that satisfies Condition 1 and Conditions 5–7.

4.2. Propositions not involving p

Throughout what follows, σ_1 and σ_2 are any sentences in $\mathcal{L}$.

PROPOSITION 2. *If $\sigma_\alpha \in \{\sigma_{ab}, \sigma_a, \sigma_b, \sigma_\emptyset\}$ then $\sigma_1.\sigma_\alpha$ is L-false or L-equivalent to σ_α.*

Proof. Suppose $\sigma_\alpha \in \{\sigma_{ab}, \sigma_a, \sigma_b, \sigma_\emptyset\}$. Then there is only one admissible model for $\mathcal{L}$ in which σ_α is true. For example, σ_{ab} is true in the model that assigns truth to 'Fa' and 'Fb' and falsity to 'Ga' and 'Gb'; it is false in the other eight admissible models. Let M be the admissible model in which σ_α is true. If σ_1 is true in M then $\sigma_1.\sigma_\alpha$ is true in M and false in every other admissible model, so $\sigma_1.\sigma_\alpha$ is L-equivalent to σ_α. If σ_1 is false in M then $\sigma_1.\sigma_\alpha$ is false in every admissible model and hence L-false. So $\sigma_1.\sigma_\alpha$ is either L-false or L-equivalent to σ_α.

PROPOSITION 3. *If $r(\sigma_1) \neq \emptyset$ then every disjunction of all elements of $r(\sigma_1)$ is L-equivalent to $\sigma_1.\sigma_*$.*

Proof. Suppose $r(\sigma_1) \neq \emptyset$ and let σ_{1r} be a disjunction of all elements of $r(\sigma_1)$. Let '$\Leftrightarrow$' denote L-equivalence. Then

$$\begin{aligned}
\sigma_{1r} \;&\Leftrightarrow\; (\sigma_1.\sigma_{ab}) \vee (\sigma_1.\sigma_a) \vee (\sigma_1.\sigma_b) \vee (\sigma_1.\sigma_\emptyset),\\
&\qquad \text{by Definition 2 and Proposition 2}\\
&\Leftrightarrow\; \sigma_1.(\sigma_{ab} \vee \sigma_a \vee \sigma_b \vee \sigma_\emptyset)\\
&\Leftrightarrow\; \sigma_1.\sigma_*, \text{ since } \sigma_* \Leftrightarrow \sigma_{ab} \vee \sigma_a \vee \sigma_b \vee \sigma_\emptyset.
\end{aligned}$$

PROPOSITION 4. *$r(\sigma_1.\sigma_2) \cup r(\sim\sigma_1.\sigma_2) = r(\sigma_2)$ and $r(\sigma_1.\sigma_2) \cap r(\sim\sigma_1.\sigma_2) = \emptyset$.*

Proof. (i) Suppose $\sigma_\alpha \in r(\sigma_1.\sigma_2)$. Then by Definition 2, $\sigma_1.\sigma_2.\sigma_\alpha$ is not L-false. It follows that $\sigma_2.\sigma_\alpha$ is not L-false and hence $\sigma_\alpha \in r(\sigma_2)$. Thus $r(\sigma_1.\sigma_2) \subset r(\sigma_2)$. Similarly, $r(\sim\sigma_1.\sigma_2) \subset r(\sigma_2)$ and so

$$r(\sigma_1.\sigma_2) \cup r(\sim\sigma_1.\sigma_2) \subset r(\sigma_2). \tag{1}$$

(ii) Suppose $\sigma_\alpha \in r(\sigma_2)$. Then by Definition 2, $\sigma_2.\sigma_\alpha$ is not L-false. It follows that not both of $\sigma_1.\sigma_2.\sigma_\alpha$ and $\sim\sigma_1.\sigma_2.\sigma_\alpha$ are L-false and hence

$$\sigma_\alpha \in r(\sigma_1.\sigma_2) \cup r(\sim\sigma_1.\sigma_2).$$

Thus

$$r(\sigma_2) \subset r(\sigma_1.\sigma_2) \cup r(\sim\sigma_1.\sigma_2). \tag{2}$$

(iii) From (1) and (2) it follows that

$$r(\sigma_1.\sigma_2) \cup r(\sim\sigma_1.\sigma_2) = r(\sigma_2).$$

(iv) Suppose $\sigma_\alpha \in r(\sigma_1.\sigma_2) \cap r(\sim\sigma_1.\sigma_2)$. Then by Definition 2, neither $\sigma_1.\sigma_2.\sigma_\alpha$ nor $\sim\sigma_1.\sigma_2.\sigma_\alpha$ is L-false. So by Proposition 2, $\sigma_1.\sigma_2.\sigma_\alpha$ and $\sim\sigma_1.\sigma_2.\sigma_\alpha$ are both L-equivalent to σ_α. Hence $\sigma_1.\sigma_2.\sigma_\alpha$ is L-equivalent to $\sim\sigma_1.\sigma_2.\sigma_\alpha$. Since these cannot both be true they must both be L-false. But we deduced that neither is L-false. This contradiction shows that the assumption from which we began is false, i.e., there is no $\sigma_\alpha \in r(\sigma_1.\sigma_2) \cap r(\sim\sigma_1.\sigma_2)$ and so

$$r(\sigma_1.\sigma_1) \cap r(\sim\sigma_1.\sigma_2) = \emptyset.$$

4.3. Necessity

This section will show that Conditions 2–4 imply Conditions 5–7. Throughout what follows, σ_1 and σ_2 are any sentences in $\mathcal{L}$ and σ_3 is any sentence in $\mathcal{L}$ that L-implies σ_*.

PROPOSITION 5. *Condition 2 implies that if σ_3 L-implies σ_1 then $p(\sigma_1, \sigma_3) = 1$.*

Proof. Assume Condition 2 holds and σ_3 L-implies σ_1. Then

$$\begin{aligned} p(\sigma_1, \sigma_3) &= p(\sigma_1, \sigma_3)\, p(\sigma_3, \sigma_3), \text{ by A2} \\ &= p(\sigma_1, \sigma_3)\, p(\sigma_3, \sigma_1.\sigma_3), \text{ by A5} \\ &= p(\sigma_1.\sigma_3, \sigma_3), \text{ by A4} \\ &= p(\sigma_3, \sigma_3), \text{ by A5} \\ &= 1, \text{ by A2.} \end{aligned}$$

PROPOSITION 6. *Condition 2 implies that if $\sigma_1.\sigma_3$ is L-false and σ_3 is not L-false then $p(\sigma_1, \sigma_3) = 0$.*

Proof. Assume Condition 2 holds, $\sigma_1.\sigma_3$ is L-false, and σ_3 is not L-false. Then σ_3 L-implies $\sim\sigma_1$ and

$$\begin{aligned} p(\sigma_1, \sigma_3) &= 1 - p(\sim\sigma_1, \sigma_3), \text{ by A3} \\ &= 1 - 1, \text{ by Proposition 5} \\ &= 0. \end{aligned}$$

PROPOSITION 7. *Condition 2 implies that $p(\sigma_1, \sigma_3) = p(\sigma_1.\sigma_3, \sigma_3)$.*

Proof. Assume Condition 2 holds. Then

$$\begin{aligned} p(\sigma_1, \sigma_3) &= p(\sigma_1, \sigma_3) p(\sigma_3, \sigma_1.\sigma_3), \text{ by Proposition 5} \\ &= p(\sigma_1.\sigma_3, \sigma_3), \text{ by A4.} \end{aligned}$$

PROPOSITION 8. *Condition 2 implies that if $\sigma_1.\sigma_3$ is L-equivalent to $\sigma_2.\sigma_3$ then $p(\sigma_1, \sigma_3) = p(\sigma_2, \sigma_3)$.*

Proof. Assume Condition 2 holds and $\sigma_1.\sigma_3$ is L-equivalent to $\sigma_2.\sigma_3$. Then

$$\begin{aligned} p(\sigma_1, \sigma_3) &= p(\sigma_1.\sigma_3, \sigma_3), \text{ by Proposition 7} \\ &= p(\sigma_2.\sigma_3, \sigma_3), \text{ by A5} \\ &= p(\sigma_2, \sigma_3), \text{ by Proposition 7.} \end{aligned}$$

PROPOSITION 9. *Condition 2 implies that if σ_3 L-implies $\sim(\sigma_1.\sigma_2)$ and σ_3 is not L-false then $p(\sigma_1 \vee \sigma_2, \sigma_3) = p(\sigma_1, \sigma_3) + p(\sigma_2, \sigma_3)$.*

Proof. Assume Condition 2 holds, σ_3 L-implies $\sim(\sigma_1.\sigma_2)$, and σ_3 is not L-false. If $(\sigma_1 \vee \sigma_2).\sigma_3$ is not L-false then

$$\begin{aligned} p(\sigma_1 \vee \sigma_2, \sigma_3) &= p(\sigma_1 \vee \sigma_2, \sigma_3)[p(\sigma_1, (\sigma_1 \vee \sigma_2).\sigma_3) + \\ &\quad p(\sim\sigma_1, (\sigma_1 \vee \sigma_2).\sigma_3)], \text{ by A3} \\ &= p((\sigma_1 \vee \sigma_2).\sigma_1, \sigma_3) + p((\sigma_1 \vee \sigma_2).\sim\sigma_1, \sigma_3), \text{ by A4} \\ &= p(\sigma_1, \sigma_3) + p(\sigma_2.\sim\sigma_1, \sigma_3), \text{ by A5} \\ &= p(\sigma_1, \sigma_3) + p(\sigma_2, \sigma_3), \text{ by Proposition 8.} \end{aligned}$$

If $(\sigma_1 \vee \sigma_2).\sigma_3$ is L-false then $\sigma_1.\sigma_3$ and $\sigma_2.\sigma_3$ are also L-false and it follows from Proposition 6 that

$$p(\sigma_1 \vee \sigma_2, \sigma_3) = p(\sigma_1, \sigma_3) = p(\sigma_2, \sigma_3) = 0.$$

So in either case, $p(\sigma_1 \vee \sigma_2, \sigma_3) = p(\sigma_1, \sigma_3) + p(\sigma_2, \sigma_3)$.

PROPOSITION 10. *Conditions 2–4 imply Condition 5.*

Proof. Assume Conditions 2–4 hold. Then

$$\begin{aligned} p(\sigma_{ab}, \sigma_*) &= p(Fa.Fb, \sigma_*), \text{ by Definition 1} \\ &= p(Fa, \sigma_*)p(Fb, Fa.\sigma_*), \text{ by A4} \\ &= 1/3, \text{ by Conditions 3 and 4.} \end{aligned} \tag{3}$$

$$\begin{aligned} p(\sigma_a, \sigma_*) &= p(Fa.Gb, \sigma_*), \text{ by Definition 1} \\ &= p(Fa, \sigma_*)p(Gb, Fa.\sigma_*), \text{ by A4} \\ &= p(Fa, \sigma_*)p(\sim Fb, Fa.\sigma_*), \text{ by Proposition 8} \\ &= p(Fa, \sigma_*)[1 - p(Fb, Fa.\sigma_*)], \text{ by A3} \\ &= 1/6, \text{ by Conditions 3 and 4.} \end{aligned} \tag{4}$$

$$\begin{aligned} p(\sigma_b, \sigma_*) &= p(Ga.Fb, \sigma_*), \text{ by Definition 1} \\ &= p((Fa.Fb) \vee (Ga.Fb), \sigma_*) - p(Fa.Fb, \sigma_*), \end{aligned}$$

$$\begin{aligned} & \text{by Proposition 9} \\ = \ & p(Fb, \sigma_*) - p(Fa.Fb, \sigma_*), \text{ by Proposition 8} \\ = \ & p(Fb, \sigma_*) - p(\sigma_{ab}, \sigma_*), \text{ by Definition 1} \\ = \ & 1/6, \text{ by Condition 3 and (3).} \end{aligned} \tag{5}$$

$$\begin{aligned} p(\sigma_\emptyset, \sigma_*) \ &= \ 1 - p(\sim\sigma_\emptyset, \sigma_*), \text{ by A3} \\ &= \ 1 - p(\sigma_{ab} \vee \sigma_a \vee \sigma_b, \sigma_*), \text{ by Proposition 8} \\ &= \ 1 - p(\sigma_{ab}, \sigma_*) - p(\sigma_a, \sigma_*) - p(\sigma_b, \sigma_*), \\ & \quad \text{by Proposition 9} \\ &= \ 1/3, \text{ by (3), (4), and (5).} \end{aligned} \tag{6}$$

Identities (3)–(6) are what Condition 5 asserts.

PROPOSITION 11. *Condition 2 implies Condition 6.*

Proof. Assume Condition 2 holds. If $\sigma_1.\sigma_*$ is L-false then

$$\begin{aligned} p(\sigma_1, \sigma_*) \ &= \ 0, \text{ by Proposition 6} \\ &= \ \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_1), \text{ since } r(\sigma_1) = \emptyset. \end{aligned}$$

If $\sigma_1.\sigma_*$ is not L-false then $r(\sigma_1) \neq \emptyset$. If σ_{1r} is a disjunction of all elements of $r(\sigma_1)$ we then have

$$\begin{aligned} p(\sigma_1, \sigma_*) \ &= \ p(\sigma_1.\sigma_*, \sigma_*), \text{ by Proposition 7} \\ &= \ p(\sigma_{1r}, \sigma_*), \text{ by A5 and Proposition 3} \\ &= \ \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_*), \text{ by Proposition 9.} \end{aligned}$$

Thus

$$p(\sigma_1, \sigma_*) = \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_*)$$

for all sentences σ_1 in $\mathcal{L}$, and in particular for all sentences in $\mathcal{L}$ other than σ_{ab}, σ_a, σ_b, and $\sigma_\emptyset$. This is what Condition 6 asserts.

PROPOSITION 12. *Conditions 2–4 imply Condition 7.*

Proof. Assume Conditions 2–4 hold. Then

$$\begin{aligned} p(\sigma_1.\sigma_3, \sigma_*) \ &= \ p(\sigma_3.\sigma_1, \sigma_*), \text{ by A5} \\ &= \ p(\sigma_3, \sigma_*)p(\sigma_1, \sigma_3.\sigma_*), \text{ by A4} \\ &= \ p(\sigma_3, \sigma_*)p(\sigma_1, \sigma_3), \text{ by A5.} \end{aligned}$$

Hence

$$p(\sigma_1, \sigma_3) = \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ if } p(\sigma_3, \sigma_*) \neq 0. \tag{7}$$

Now suppose $p(\sigma_3, \sigma_*) = 0$. Then

$$\begin{aligned} \sum_{\sigma \in r(\sigma_3)} p(\sigma, \sigma_*) &= p(\sigma_3, \sigma_*), \text{ by Proposition 11} \\ &= 0, \text{ by assumption.} \end{aligned}$$

By Proposition 10 and Definition 2, $p(\sigma, \sigma_*) > 0$ for all $\sigma \in r(\sigma_3)$, so $r(\sigma_3) = \emptyset$. Hence $\sigma_3.(\sigma_{ab} \vee \sigma_a \vee \sigma_b \vee \sigma_\emptyset)$ is L-false. Since σ_* is L-equivalent to $\sigma_{ab} \vee \sigma_a \vee \sigma_b \vee \sigma_\emptyset$, it follows that $\sigma_3.\sigma_*$ is L-false. Since σ_3 L-implies σ_*, it follows that σ_3 is L-false. An L-false sentence L-implies every sentence, hence σ_3 L-implies σ_1 and so, by Proposition 5, $p(\sigma_1, \sigma_3) = 1$. Thus

$$p(\sigma_1, \sigma_3) = 1, \text{ if } p(\sigma_3, \sigma_*) = 0. \tag{8}$$

(7) and (8) together imply Condition 7.

PROPOSITION 13. *Conditions 2–4 imply Conditions 5–7.*

Proof. Propositions 10, 11, and 12.

4.4. Sufficiency

This section will show that Conditions 5–7 imply Conditions 2–4. As before, σ_1 and σ_2 are here any sentences in $\mathcal{L}$ and σ_3 is any sentence in $\mathcal{L}$ that L-implies σ_*.

PROPOSITION 14. *Condition 6 implies*

$$p(\sigma_1, \sigma_*) = \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_*). \tag{9}$$

Proof. Condition 6 states that (9) holds if σ_1 is not σ_{ab}, σ_a, σ_b, or $\sigma_\emptyset$. If σ_1 is σ_{ab}, σ_a, σ_b, or $\sigma_\emptyset$ then $r(\sigma_1) = \{\sigma_1\}$ and so (9) holds trivially. Hence Condition 6 implies that (9) holds for all σ_1.

PROPOSITION 15. *Conditions 5–7 imply*

$$p(\sigma_1, \sigma_3) = \left\{ \begin{array}{cc} \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)} & \textit{if } p(\sigma_3, \sigma_*) \neq 0 \\ 1 & \textit{if } p(\sigma_3, \sigma_*) = 0 \end{array} \right\}. \tag{10}$$

Proof. Assume Conditions 5–7. If $\sigma_3 \neq \sigma_*$ then (10) holds by Condition 7. Also

$$\begin{aligned} p(\sigma_*, \sigma_*) &= \sum_{\sigma \in r(\sigma_*)} p(\sigma, \sigma_*), \text{ by Proposition 14} \\ &= p(\sigma_{ab}, \sigma_*) + p(\sigma_a, \sigma_*) + p(\sigma_b, \sigma_*) + p(\sigma_\emptyset, \sigma_*), \quad (11) \\ &\quad \text{by Definition 2} \\ &= 1, \text{ by Condition 5.} \quad (12) \end{aligned}$$

So for the case where $\sigma_3 = \sigma_*$ we have $p(\sigma_3, \sigma_*) \neq 0$ and

$$\begin{aligned} p(\sigma_1, \sigma_3) &= p(\sigma_1, \sigma_*), \text{ since } \sigma_3 = \sigma_* \\ &= \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_*), \text{ by Proposition 14} \\ &= \sum_{\sigma \in r(\sigma_1.\sigma_*)} p(\sigma, \sigma_*), \text{ since } r(\sigma_1.\sigma_*) = r(\sigma_1) \\ &= p(\sigma_1.\sigma_*, \sigma_*), \text{ by Proposition 14} \\ &= \frac{p(\sigma_1.\sigma_*, \sigma_*)}{p(\sigma_*, \sigma_*)}, \text{ by (12)} \\ &= \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ since } \sigma_3 = \sigma_*. \end{aligned}$$

Hence (10) holds for all σ_1 and σ_3.

PROPOSITION 16. *Conditions 5–7 imply A1.*

Proof. Assume Conditions 5–7. If $p(\sigma_3, \sigma_*) \neq 0$ then

$$\begin{aligned} p(\sigma_1, \sigma_3) &= \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ by Proposition 15} \\ &= \frac{\sum_{\sigma \in r(\sigma_1.\sigma_3)} p(\sigma, \sigma_*)}{\sum_{\sigma \in r(\sigma_3)} p(\sigma, \sigma_*)}, \text{ by Proposition 14} \\ &\geq 0, \text{ by Definition 2 and Condition 5.} \end{aligned}$$

If $p(\sigma_3, \sigma_*) = 0$ then $p(\sigma_1, \sigma_3) = 1 \geq 0$, by Proposition 15. Hence $p(\sigma_1, \sigma_3) \geq 0$ for all σ_1 and σ_3, which is what A1 asserts.

PROPOSITION 17. *Conditions 5–7 imply A5.*

Proof. Assume Conditions 5–7. Let σ_1' and σ_3' be L-equivalent to σ_1 and σ_3 respectively.

$$p(\sigma_1', \sigma_*) = \sum_{\sigma \in r(\sigma_1')} p(\sigma, \sigma_*), \text{ by Proposition 14}$$

$$= \sum_{\sigma \in r(\sigma_1)} p(\sigma, \sigma_*), \text{ since } r(\sigma_1) = r(\sigma_1')$$
$$= p(\sigma_1, \sigma_*), \text{ by Proposition 14.} \quad (13)$$

If $p(\sigma_3, \sigma_*) \neq 0$ then it follows from (13) that $p(\sigma_3', \sigma_*) \neq 0$ and so

$$\begin{aligned} p(\sigma_1', \sigma_3') &= \frac{p(\sigma_1'.\sigma_3', \sigma_*)}{p(\sigma_3', \sigma_*)}, \text{ by Proposition 15} \\ &= \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ by (13)} \\ &= p(\sigma_1, \sigma_3), \text{ by Proposition 15.} \end{aligned}$$

If $p(\sigma_3, \sigma_*) = 0$ then it follows from (13) that $p(\sigma_3', \sigma_*) = 0$ and so

$$\begin{aligned} p(\sigma_1', \sigma_3') &= 1, \text{ by Proposition 15} \\ &= p(\sigma_1, \sigma_3), \text{ by Proposition 15.} \end{aligned}$$

In either case, $p(\sigma_1', \sigma_3') = p(\sigma_1, \sigma_3)$, as A5 asserts.

PROPOSITION 18. *Conditions 5–7 imply A2.*

Proof. Assume Conditions 5–7. If $p(\sigma_3, \sigma_*) \neq 0$ then

$$\begin{aligned} p(\sigma_3, \sigma_3) &= \frac{p(\sigma_3.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ by Proposition 15} \\ &= \frac{p(\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ by Proposition 17} \\ &= 1. \end{aligned}$$

If $p(\sigma_3, \sigma_*) = 0$ then $p(\sigma_3, \sigma_3) = 1$ by Condition 7. In either case, $p(\sigma_3, \sigma_3) = 1$, as A2 asserts.

PROPOSITION 19. *Conditions 5–7 imply A3.*

Proof. Assume Conditions 5–7 hold and σ_3 is not L-false. It is a standing assumption that σ_3 L-implies σ_*, so $\sigma_3.\sigma_*$ is not L-false and hence $r(\sigma_3) \neq \emptyset$. Thus

$$\begin{aligned} p(\sigma_3, \sigma_*) &= \sum_{\sigma \in r(\sigma_3)} p(\sigma, \sigma_*), \text{ by Proposition 14} \\ &> 0, \text{ by Condition 5 and } r(\sigma_3) \neq \emptyset. \end{aligned} \quad (14)$$

$$\begin{aligned} & p(\sigma_1, \sigma_3) + p(\sim\sigma_1, \sigma_3) \\ &= \frac{p(\sigma_1.\sigma_3, \sigma_*) + p(\sim\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}, \text{ by (14) and Proposition 15} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sum_{\sigma\in r(\sigma_1.\sigma_3)} p(\sigma,\sigma_*) + \sum_{\sigma\in r(\sim\sigma_1.\sigma_3)} p(\sigma,\sigma_*)}{p(\sigma_3,\sigma_*)}, \text{ by Proposition 14}\\
&= \frac{\sum_{\sigma\in r(\sigma_3)} p(\sigma,\sigma_*)}{p(\sigma_3,\sigma_*)}, \text{ by Proposition 4}\\
&= \frac{p(\sigma_3,\sigma_*)}{p(\sigma_3,\sigma_*)}, \text{ by Proposition 14}\\
&= 1.
\end{aligned}
$$

This is what A3 asserts.

PROPOSITION 20. *Conditions 5–7 imply that if σ_1 L-implies σ_2 and $p(\sigma_2,\sigma_*)=0$ then $p(\sigma_1,\sigma_*)=0$.*

Proof. Assume Conditions 5–7 hold, σ_1 L-implies σ_2, and $p(\sigma_2,\sigma_*)=0$. Then

$$
\begin{aligned}
p(\sigma_1,\sigma_*) &= \sum_{\sigma\in r(\sigma_1)} p(\sigma,\sigma_*), \text{ by Proposition 14}\\
&\le \sum_{\sigma\in r(\sigma_2)} p(\sigma,\sigma_*), \text{ since } r(\sigma_1)\subset r(\sigma_2) \text{ and}\\
&\quad\ p(\sigma,\sigma_*)>0 \text{ for all } \sigma\in r(\sigma_2)\\
&= p(\sigma_2,\sigma_*), \text{ by Proposition 14}\\
&= 0, \text{ by assumption.}
\end{aligned}
$$

By Proposition 16, $p(\sigma_1,\sigma_*)\ge 0$. Hence $p(\sigma_1,\sigma_*)=0$.

PROPOSITION 21. *Conditions 5–7 imply A4.*

Proof. Assume Conditions 5–7. We consider three cases.

Case 1: $p(\sigma_1.\sigma_3,\sigma_*)\neq 0$. Then it follows from Proposition 20 that $p(\sigma_3,\sigma_*)\neq 0$ and so

$$
\begin{aligned}
p(\sigma_1.\sigma_2,\sigma_3) &= \frac{p((\sigma_1.\sigma_2).\sigma_3,\sigma_*)}{p(\sigma_3,\sigma_*)}, \text{ by Proposition 15}\\
&= \frac{p(\sigma_2.(\sigma_1.\sigma_3),\sigma_*)}{p(\sigma_3,\sigma_*)}, \text{ by Proposition 17}\\
&= \frac{p(\sigma_1.\sigma_3,\sigma_*)}{p(\sigma_3,\sigma_*)}\,\frac{p(\sigma_2.(\sigma_1.\sigma_3),\sigma_*)}{p(\sigma_1.\sigma_3,\sigma_*)}, \text{ by arithmetic}\\
&= p(\sigma_1,\sigma_3)p(\sigma_2,\sigma_1.\sigma_3), \text{ by Proposition 15.}
\end{aligned}
$$

Case 2: $p(\sigma_1.\sigma_3,\sigma_*)=0$ and $p(\sigma_3,\sigma_*)\neq 0$. Then

$$
p(\sigma_1.\sigma_2,\sigma_3) = \frac{p((\sigma_1.\sigma_2).\sigma_3,\sigma_*)}{p(\sigma_3,\sigma_*)}, \text{ by Proposition 15}
$$

$$
\begin{aligned}
&= 0, \text{ by Proposition 20} \\
&= 0.p(\sigma_2, \sigma_1.\sigma_3), \text{ by arithmetic} \\
&= \frac{p(\sigma_1.\sigma_3, \sigma_*)}{p(\sigma_3, \sigma_*)}.p(\sigma_2, \sigma_1.\sigma_3), \\
&\quad \text{by the assumptions of this case} \\
&= p(\sigma_1, \sigma_3)p(\sigma_2, \sigma_1.\sigma_3), \text{ by Proposition 15.}
\end{aligned}
$$

Case 3: $p(\sigma_3, \sigma_*) = 0$. Then $p(\sigma_1.\sigma_3, \sigma_*) = 0$ by Proposition 20 and so

$$
\begin{aligned}
p(\sigma_1.\sigma_2, \sigma_3) &= 1, \text{ by Proposition 15} \\
&= p(\sigma_1, \sigma_3)p(\sigma_2, \sigma_1.\sigma_3), \text{ by Proposition 15.}
\end{aligned}
$$

So in every case, $p(\sigma_1.\sigma_2, \sigma_3) = p(\sigma_1, \sigma_3)p(\sigma_2, \sigma_1.\sigma_3)$, as A4 asserts.

PROPOSITION 22. *Conditions 5–7 imply Condition 2.*
Proof. Propositions 16–19 and 21.

PROPOSITION 23. *Conditions 5–7 imply Condition 3.*
Proof. Assume Conditions 5–7. Then

$$
\begin{aligned}
p(Fa, \sigma_*) &= p(\sigma_{ab} \vee \sigma_a, \sigma_*), \text{ by Propositions 22 and 8} \\
&= p(\sigma_{ab}, \sigma_*) + p(\sigma_a, \sigma_*), \text{ by Propositions 22 and 9} \\
&= 1/2, \text{ by Condition 5.}
\end{aligned}
$$

Similarly, $p(Fb, \sigma_*) = p(\sigma_{ab}, \sigma_*) + p(\sigma_b, \sigma_*) = 1/2$.

PROPOSITION 24. *Conditions 5–7 imply Condition 4.*
Proof. Assume Conditions 5–7. Then

$$
\begin{aligned}
p(Fa.\sigma_*, \sigma_*) &= p(Fa, \sigma_*), \text{ by Propositions 22 and 8} \\
&= 1/2, \text{ by Proposition 23.} \qquad (15)
\end{aligned}
$$

So

$$
\begin{aligned}
p(Fb, Fa.\sigma_*) &= \frac{p(Fb.(Fa.\sigma_*), \sigma_*)}{p(Fa.\sigma_*, \sigma_*)}, \text{ by Proposition 15} \\
&= 2p(Fb.(Fa.\sigma_*), \sigma_*), \text{ by (15)} \\
&= 2p(Fa.Fb, \sigma_*), \text{ by Propositions 22 and 8} \\
&= 2p(\sigma_{ab}, \sigma_*), \text{ by Definition 1} \\
&= 2/3, \text{ by Condition 5.}
\end{aligned}
$$

PROPOSITION 25. *Conditions 5–7 imply Conditions 2–4.*

Proof. Propositions 22–24.

This completes the proof that I outlined at the beginning of Section 4.

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